POINCARÉ SERIES FOR LOCAL UNITARY INVARIANTS OF MIXED STATES OF THE QUBIT-QUTRIT SYSTEM

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Abstract. We consider the mixed states of the bipartite quantum system with the first party a qubit and the second a qutrit. The Hilbert space of this system is the tensor product $\mathcal{H} = \mathbf{C}^2 \otimes \mathbf{C}^3$ and the group of local unitary transformations, ignoring the overall phase factor, is the direct product $G = \mathrm{SU}(2) \times \mathrm{SU}(3)$. Let $\mathcal{P}$ be the algebra of real polynomial functions on the affine space of all hermitian operators of trace 1 on $\mathcal{H}$. The polynomials $f \in \mathcal{P}$ which are invariant under G form the subalgebra $\mathcal{P}^G \subseteq \mathcal{P}$. We compute the simply graded Poincaré series of this subalgebra and construct several low degree invariants.

1. Introduction

Let $\mathbf{C}^n$ be the Hilbert space (of column vectors) with the inner product $\langle x_1, x_2 \rangle = x_1^\dagger x_2$ and M_n the algebra of complex $n \times n$ matrices. For $X, Y \in M_n$, their inner product is defined by $\langle X_1, X_2 \rangle = \mathrm{tr}(X_1^\dagger X_2)$. Let $\mathfrak{su}(n)$ denote the Lie algebra of $\mathrm{SU}(n)$, it consists of all traceless skew-hermitian matrices in M_n.

We consider the bipartite quantum system with the first party a qubit, $\mathbf{C}^2$, and the second one a qutrit, $\mathbf{C}^3$. It will be convenient to identify $\mathbf{C}^6$ with the tensor product $\mathcal{H} = \mathbf{C}^2 \otimes_{\mathbf{C}} \mathbf{C}^3$ and M_6 with $M_2 \otimes_{\mathbf{C}} M_3$. For $X_1, X_2 \in M_2$ and $Y_1, Y_2 \in M_3$, the inner product of $X_1 \otimes Y_1$ and $X_2 \otimes Y_2$ is given by

$$\langle X_1 \otimes Y_1, X_2 \otimes Y_2 \rangle = \langle X_1, X_2 \rangle \cdot \langle Y_1, Y_2 \rangle.$$

It follows that $(X \otimes Y)^\dagger = X^\dagger \otimes Y^\dagger$ and, in particular, the tensor product of hermitian matrices is again a hermitian matrix.

The space of hermitian matrices $H_n \subseteq M_n$ is the direct sum of the $1D$ real space spanned by the identity matrix, I_n, and the space $H_{n,0} = i\mathfrak{su}(n)$. The space $H_{6,0}$ is the direct sum of three real subspaces: $V_1 = H_{2,0} \otimes I_3$, $V_2 = I_2 \otimes H_{3,0}$, and $V_3 = H_{2,0} \otimes H_{3,0}$. Any mixed state, ρ, of this quantum system can be written uniquely as the sum of four components;

$$\rho = \frac{1}{6} I_6 + X \otimes I_3 + I_2 \otimes Y + Z, \tag{1.1}$$

where $X \in H_{2,0}$, $Y \in H_{3,0}$, and $Z \in V_3$. If we fix a basis $\{E_1, E_2, E_3\}$ of $H_{2,0}$, then Z can be written uniquely as

$$Z = \sum_{k=1}^{3} E_k \otimes Y_k, \quad Y_k \in H_{3,0}. \tag{1.2}$$

For instance, we could take the basis consisting of the three Pauli matrices.

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Denote by G the group of local unitary transformations, $\mathrm{SU}(2)\times\mathrm{SU}(3)$, where we ignore the overall phase factor. Note that G acts on M_6 in the usual manner: $(g,Z)\to gZg^\dagger$. It stabilizes the real subspaces H_6 and $H_{6,0}=V_1+V_2+V_3$. Moreover, each of the subspaces V_1, V_2, V_3 is a simple G-module.

Let $\mathcal{P}$ denote the algebra of real valued polynomial functions on $H_{6,0}$ and $\mathcal{P}^G$ the subalgebra of G-invariant functions. Note that $\mathcal{P}^G$ inherits the $\mathbf{Z}$-gradation from $\mathcal{P}$. The homogeneous polynomials of $\mathcal{P}^G$ may be used to construct measures of entanglement of this quantum system.

The main objective of this note is to determine the Poincaré series (also known as the Hilbert series) of $\mathcal{P}^G$. The knowledge of the algebra $\mathcal{P}^G$ is important because of the following well known fact: Two states, say ρ_1 and ρ_2, belong to different G-orbits iff there exists $f\in\mathcal{P}^G$ such that $f(\rho_1)\neq f(\rho_2)$. Due to the direct decomposition of $H_{6,0}$ into three G-invariant subspaces, $\mathcal{P}$ also admits a G-invariant $\mathbf{Z}^3$-gradation. It induces $\mathbf{Z}^3$-gradation on $\mathcal{P}^G$, to which we refer as the multigradation. So far we did not succeed to compute the multigraded Poincaré series of $\mathcal{P}^G$.

The case of two qubits has been considered by M. Grassl et al. [6], Y. Makhlin [10], and also by the author [5]. The local invariants for pure states of more qubits have been considered in several recent papers: see [1, 3, 9, 12, 13] for 3 qubits, [2, 7, 9] for 4 qubits, and [8] for 5 qubits.

2. The Poincaré series for the $2\otimes 3$ case

Going back to our case, the Poncaré series $P(t)$ is given by

$$P(t)=\sum_{d=0}^{\infty}\dim(\mathcal{P}_d^G)t^d,$$

where $\mathcal{P}_d^G$ is the space of homogeneous polynomial invariants of degree d. This Poincaré series is a rational function of the variable t. It can be computed by using the well-known Molien–Weyl formula. Following the recipe from [4], we obtain that

$$P(t)=\frac{1}{(2\pi i)^3}\int_{|z|=1}\int_{|y|=1}\int_{|x|=1}\varphi(x,y,z,t)\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}\frac{\mathrm{d}z}{z},$$

where

$$\varphi(x,y,z,t)=\frac{(1-x^{-1})(1-y^{-1})(1-z^{-1})(1-y^{-1}z^{-1})}{\psi(x,y,z,t)}$$

and

$$\begin{aligned}\psi\ =\ &(1-t)^5(1-tx)^3(1-tx^{-1})^3(1-ty)^2(1-tz)^2(1-tyz)^2\\ &(1-ty^{-1})^2(1-tz^{-1})^2(1-ty^{-1}z^{-1})^2(1-txy)(1-txz)(1-txyz)\\ &(1-txy^{-1})(1-txz^{-1})(1-txy^{-1}z^{-1})(1-tx^{-1}y)(1-tx^{-1}z)(1-tx^{-1}yz)\\ &(1-tx^{-1}y^{-1})(1-tx^{-1}z^{-1})(1-tx^{-1}y^{-1}z^{-1}).\end{aligned}$$

The integrations are over the unit circle in the counterclockwise direction, and one should assume that $|t|<1$. The computation of these integrals was performed by using Maple [11]. Our main result is the following.

Theorem 2.1. *The Poincaré series of $\mathcal{P}^G$ is the rational function $P(t)$ whose numerator $N(t)$ and denominator $D(t)$ are given by*

$$\begin{aligned} N &= 1+t-2\,t^3+2\,t^4+13\,t^5+50\,t^6+102\,t^7+216\,t^8+422\,t^9+874\,t^{10} \\ &\quad +1691\,t^{11}+3305\,t^{12}+6037\,t^{13}+10779\,t^{14}+18312\,t^{15}+30318\,t^{16} \\ &\quad +48209\,t^{17}+74858\,t^{18}+112294\,t^{19}+164391\,t^{20}+233394\,t^{21}+323332\,t^{22} \\ &\quad +435113\,t^{23}+571671\,t^{24}+730844\,t^{25}+912641\,t^{26}+1110648\,t^{27} \\ &\quad +1321048\,t^{28}+1532768\,t^{29}+1739258\,t^{30}+1926469\,t^{31}+2087251\,t^{32} \\ &\quad +2208470\,t^{33}+2286037\,t^{34}+2311126\,t^{35}+2286037\,t^{36}+\cdots \\ &\quad +2\,t^{66}-2\,t^{67}+t^{69}+t^{70}, \\ D &= (1+t)\left(1-t^2\right)^3\left(1-t^3\right)^6\left(1-t^4\right)^5\left(1-t^5\right)^4\left(1-t^6\right)^3\left(1-t^7\right)^2\left(1-t^8\right). \end{aligned}$$

(As N is a palindromic polynomial of degree 70, it suffices to write its coefficients up to degree 35.)

The Taylor expansion of $P(t)$ begins with the following terms

$$\begin{aligned} P(t) &= 1+3\,t^2+4\,t^3+15\,t^4+25\,t^5+90\,t^6+170\,t^7+489\,t^8+1059\,t^9 \\ &\quad +2600\,t^{10}+5641\,t^{11}+12872\,t^{12}+27099\,t^{13}+57990\,t^{14}+118254\,t^{15} \\ &\quad +240187\,t^{16}+472273\,t^{17}+919432\,t^{18}+1745295\,t^{19}+\cdots \end{aligned}$$

The numerator N and the denominator D are relatively prime. After multiplying them with $(1-t+t^2)(1+t^3)$, we obtain $P(t)=N^*(t)/D^*(t)$ where N^* is a palindromic polynomial with nonnegative integer coefficients

$$\begin{aligned} N^* &= 1+4\,t^4+9\,t^5+38\,t^6+69\,t^7+173\,t^8+347\,t^9+733\,t^{10}+1403\,t^{11} \\ &\quad +2796\,t^{12}+5091\,t^{13}+9286\,t^{14}+16058\,t^{15}+27208\,t^{16}+44250\,t^{17} \\ &\quad +70537\,t^{18}+108430\,t^{19}+163158\,t^{20}+238264\,t^{21}+339974\,t^{22} \\ &\quad +472130\,t^{23}+641187\,t^{24}+848615\,t^{25}+1098643\,t^{26}+1388741\,t^{27} \\ &\quad +1717327\,t^{28}+2075836\,t^{29}+2456389\,t^{30}+2843020\,t^{31}+3222408\,t^{32} \\ &\quad +3575226\,t^{33}+3884797\,t^{34}+4133599\,t^{35}+4308636\,t^{36}+4398377\,t^{37} \\ &\quad +4398377\,t^{38}+\cdots+38\,t^{69}+9\,t^{70}+4\,t^{71}+t^{75} \end{aligned}$$

and

$$D^* = \left(1-t^2\right)^3\left(1-t^3\right)^4\left(1-t^4\right)^5\left(1-t^5\right)^4\left(1-t^6\right)^5\left(1-t^7\right)^2\left(1-t^8\right).$$

As the space $H_{6,0} \subseteq M_6$ of traceless hermitian matrices has dimension 35 and the generic orbits of G have dimension 11, the orbit space of (mixed) states of the system has dimension 24. Hence, the maximum number of algebraically independent generators of $\mathcal{P}^G$ is 24. On the basis of the expression $P(t)=N^*(t)/D^*(t)$ and the factorization of D^*, it is quite reasonable to expect that a maximal set of algebraically independent homogeneous polynomials in $\mathcal{P}^G$ (in fact a homogeneous system of parameters) can be chosen so that three of them have degree 2, four degree 3, five degree 4, four degree 5, five degree 6, two degree 7, and one degree 8.

3. A few low degree invariants

We assume in this section that the density matrix ρ is given by Eq. (1.1) and that the component Z is given by Eq. (1.2).

From the above Taylor expansion of $P(t)$ we see that the space of quadratic invariants is 3-dimensional. The three linearly independent quadratic invariants are:

$$\begin{aligned} I_1(\rho) &= \det(X), \\ I_2(\rho) &= \operatorname{tr}(Y^2), \\ I_3(\rho) &= \operatorname{tr}(Z^2) = \sum_{k,l=1}^{3} \operatorname{tr}(E_k E_l) \operatorname{tr}(Y_k Y_l). \end{aligned}$$

The space of cubic invariants has dimension 4. The four linearly independent cubic invariants are given by

$$\begin{aligned} I_4(\rho) &= \det(Y), \\ I_5(\rho) &= \operatorname{tr}(Z^3) = \sum_{k,l,m=1}^{3} \operatorname{tr}(E_k E_l E_m) \operatorname{tr}(Y_k Y_l Y_m), \\ I_6(\rho) &= \operatorname{tr}((X \otimes Y)Z) = \sum_{k=1}^{3} \operatorname{tr}(X E_k) \operatorname{tr}(Y Y_k), \\ I_7(\rho) &= \operatorname{tr}((I_2 \otimes Y)Z^2) = \sum_{k,l=1}^{3} \operatorname{tr}(E_k E_l) \operatorname{tr}(Y Y_k Y_l). \end{aligned}$$

Of course, one has to check that each of these invariants is nonzero. Then the linear independence follows from the fact that they have different multidegrees, namely $(0,3,0)$, $(0,0,3)$, $(1,1,1)$ and $(0,1,2)$, respectively. (The first degree corresponds to the coordinates of V_1, the second to V_2 and the third to V_3.)

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